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MATHEMATICS
TARGET IIT JEE

AREA UNDER THE CURVE
&
DIFFERENTIAL EQUATION

THEORY AND EXERCISE BOOKLET

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AREA UNDER THE CURVE

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ANSWER KEY

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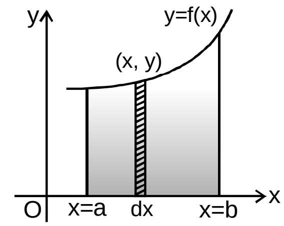
JEE Syllabus :

Application of definite integrals to the determination of areas involving simple curves, formation of ordinary differential equations, solution of homogeneous differential equations, variables separable method, linear first order differential equations.

AREA UNDER THE CURVE

A. AREA BY VERTICAL STRIPS

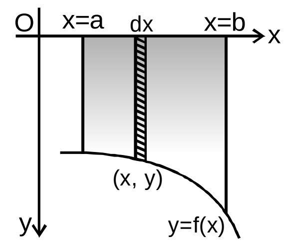
To determinent area bounded by curve y = f(x), the x-axis and the ordinates at x = a & x = b is



Case-I : If y = f(x) lies completely above the x-axis

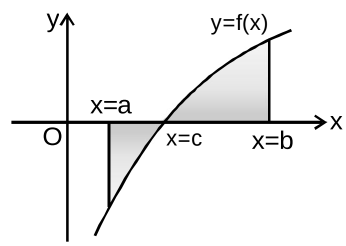
i.e. $A = \int_a^b f(x)dx = \int_a^b ydx$

Case-II : If y = f(x) lies completely below the x-axis then A is negative. The convention is to consider the magnitude only



i.e. $A = \left| \int_a^b f(x)\,dx \right| = \left| \int_a^b ydx \right|$

Case-III : If y = f(x) cuts the x-axis at x = c $\in$ (a, b)



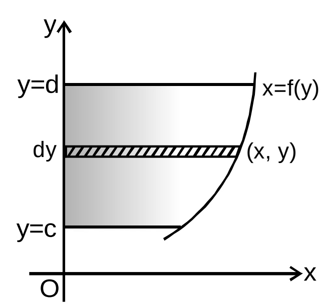
i.e. $A = \left| \int_a^c f(x)\,dx \right| + \int_c^b f(x)\,dx$

Ex.1 Find the area bounded by y = $\sec^2 x$, x = $\frac{\pi}{6}$, x = $\frac{\pi}{3}$ & x-axis

Sol. Area bounded = $\int_{\pi/6}^{\pi/3} ydx = \int_{\pi/6}^{\pi/3} \sec^2 dx = [\tan x]_{\pi/6}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{6} = \sqrt{3} - \frac{1}{\sqrt{3}} = \frac{2}{\sqrt{3}}$ sq. units

B. AREA BY HORIZONTAL STRIPS

To determine area bounded by the curve x = f(y), the y-axis and abscissa at y = c & y = d is

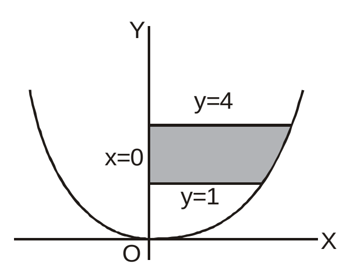


i.e. $A = \int_c^d f(y)dy = \int_c^d xdy$

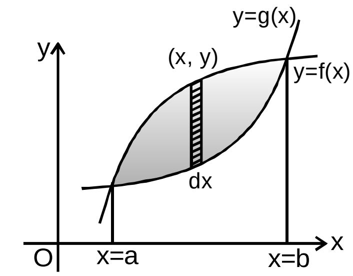
Ex.2 Find the area in the first quadrant bounded by y = $4x^2$, x = 0, y = 1 and y = 4.

Sol. The required area = $\int_1^4 xdy = \int_1^4 \frac{\sqrt{y}}{2} dy = \frac{1}{2}\left[\frac{2}{3} y^{3/2}\right]_1^4$

$= \frac{1}{3}[4^{3/2} - 1] = \frac{1}{3}[8 - 1] = \frac{7}{3} = 2\frac{1}{3}$ sq. units



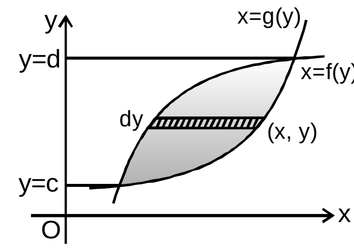
C. AREA ENCLOSED BETWEEN TWO CURVES



Case-I : (By vertical strips)

Area between the curves y = f(x) & y = g(x) between the ordinates at x = a & x = b is

i.e. $A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$



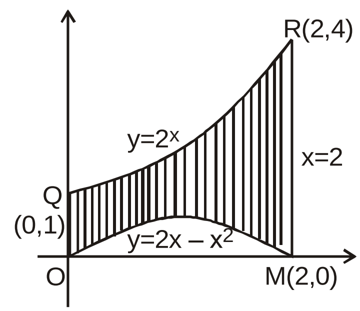
Case-II : (By horizontal strips)

Area between the curves x = f(y) & x = g(y) between the ordinates at y = c & y = d is

i.e. $A = \int_c^d f(y)dy - \int_c^d g(y)dy = \int_c^d [f(y) - g(y)]dy$

Ex.3 Compute the area of the figure bounded by the straight lines x = 0, x = 2 and the curves $y=2^x$, $y=2x - x^2$

Sol. Figure is self-explanatory $y = 2^x$, $(x - 1)^2 = -(y - 1)$



The required area = $\int_0^2 (y_1 - y_2)dx$

where $y_1 = 2^x$ and $y_2 = 2x - x^2$ = $\int_0^2 (2^x - 2x + x^2)dx$

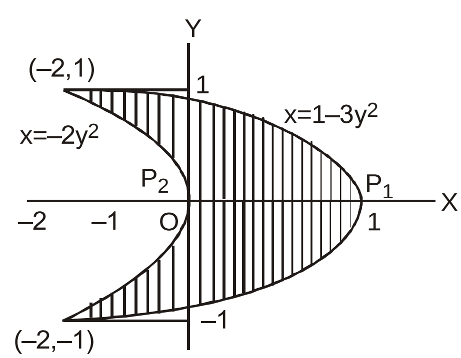
$= \left[\frac{2^x}{\ln 2} - x^2 + \frac{1}{3}x^3\right]_0^2 = \left(\frac{4}{\ln 2} - 4 + \frac{8}{3}\right) - \frac{1}{\ln 2} = \frac{3}{\ln 2} - \frac{4}{3}$ sq. units.

Ex.4 Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$

Sol. Solving the equation $x = -2y^2$, $x = 1 - 3y^2$ we find that ordinates of the points of intersection of the two curves as $y_1 = -1$, $y_2 = 1$

The points are (–2, –1) and (–2, 1)

The required area



$2\int_0^1 (x_1 - x_2)dy = 2\int_0^1 [(1 - 3y^2) - (-2y^2)]dy$

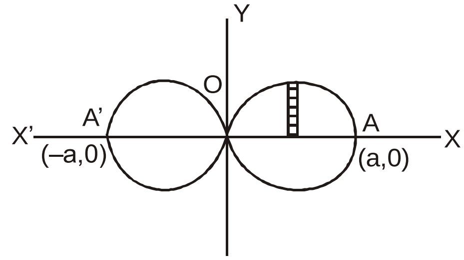
$= 2\int_0^1 (1 - y^2)dy = 2\left[y - \frac{y^3}{3}\right]_0^1 = \frac{4}{3}$ sq. units

Ex.5 Find the area of a loop as well as the whole area of the curve $a^2y^2 = x^2(a^2 - x^2)$.

Sol. The curve is symmetrical about both the axes. It cuts x-axis at (0, 0), (–a, 0), (a, 0)

Area of a loop = $2\int_0^a y\,dx = 2\int_0^a \frac{x}{a}\sqrt{a^2-x^2}\,dx$

$= -\frac{1}{a}\int_0^a \sqrt{a^2-x^2}(-2x)dx = -\frac{1}{a}\left[\frac{2}{3}(a^2-x^2)^{3/2}\right]_0^a = \frac{2}{3}a^2$



Total area = $2 \times \frac{2}{3}a^2 = \frac{4}{3}a^2$ sq. units

D. USEFUL RESULTS

(a) Whole area of the ellipse, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is π ab sq. units

(b) Area enclosed between the parabolas $y^2 = 4ax$ & $x^2 = 4$ by is $\frac{16ab}{3}$ sq. units

(c) Area included between the parabola $y^2 = 4ax$ & the line $y = mx$ is $\frac{8a^2}{3m^3}$ sq. units

E. AVERAGE VALUE OF A FUNCTION

y = f(x) w.r.t x over an interval $a \le x \le b$ is defined as : y(av) = $\frac{1}{b-a}\int_a^b f(x)dx$

Ex.6 Find the area bounded by the curve y = (x – 1) (x – 2) (x – 3) lying between the ordinates x = 0 and x = 3 and x - axis

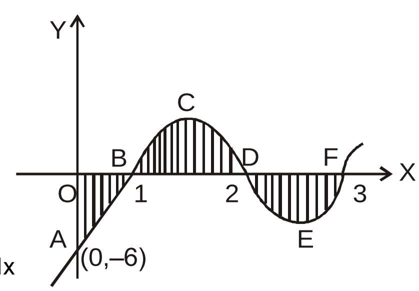
Sol. To determine the sign, we follow the usual rule as of change of sign.

y = + ve for x > 3

y = – ve for 2 < x < 3

y = +ve for 1 < x < 2

y = – ve for x < 1.



$$\int_0^3 |y|\,dx = \int_0^1 |y|\,dx + \int_1^2 |y|\,dx + \int_2^3 |y|\,dx = \int_0^1 -y\,dx + \int_1^2 y\,dx + \int_2^3 -y\,dx$$

Now let F(x) $= \int (x-1)(x-2)(x-3)dx = \int (x^3 - 6x^2 + 11x - 6)dx = \frac{1}{4}x^4 - 2x^3 + \frac{11}{2}x^2 - 6x$

$\therefore$ F(0) = 0, F(1) = $-\frac{9}{4}$, F(2) = – 2, F(3) = $-\frac{9}{4}$

Hence required Area = – [F(1) – F(0)] + [F(2) – F(1)] – [F(3) – F(2)] = $2\frac{3}{4}$ sq. units.

Ex.7 The curve $y = a\sqrt{x} + bx$ passes through the point (1, 2) and the area enclosed by the curve, the axis of x and the line x = 4 is 8 square units. Determine a, b, where a and b are positive.

Sol. The curve passes through (0, 0). Hence the limits of x are 0 to 4.

$$A = \int_0^4 y\,dx = \int_0^4 (a\sqrt{x} + bx)dx \quad \text{or} \quad 8 = \left[a.\frac{2}{3}x^{3/2} + b\frac{x^2}{2}\right]_0^4 \quad \text{or} \quad 8 = \frac{16a}{3} + 8b \qquad ...(i)$$

Again the curve passes through (1, 2) $\therefore$ 2 = a + b ...(ii)

Solving (i) and (ii), we get a = 3, b = − 1.

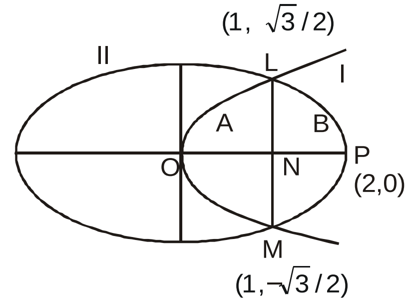
Ex.8 Find the smaller of the area bounded by the parabola $4y^2 - 3x - 8y + 7 = 0$ and the ellipse $x^2 + 4y^2 - 2x - 8y + 1 = 0$

Sol. C_1 is $4(y^2 - 2y) = 3x - 7$ or $4(y - 1)^2 = 3x - 3 = 3(x - 1)$ ...(i)

above is parabola with vertex at (1, 1)

C_2 is $(x^2 - 2x) + 4(y^2 - 2y) = -1$

or $(x - 1)^2 + 4(y - 1)^2 = -1 + 1 + 4$ or $\frac{(x-1)^2}{2^2} + \frac{(y-1)^2}{1^2} = 1$ ...(ii)



Above represents an ellipse with centre at (1, 1). Shift the origin to (1, 1) and this will not affect the magnitude of required area but will make the calculation simpler.

Thus the two curves are $4Y^2 = 3X$ and $\frac{X^2}{2^2} + \frac{Y^2}{1} = 1$ They meet at $\left(1, \pm\frac{\sqrt{3}}{2}\right)$

Required area = 2(A + B) = $2[\int Y_1 dX + \int Y_2 dX] = 2\left[\frac{\sqrt{3}}{2}\int_0^1 \sqrt{X}dX + \int_1^2 \frac{\sqrt{4 - X^2}}{2}dX\right] = \left[\frac{\sqrt{3}}{6} + \frac{2\pi}{3}\right]$

Ex.9 Find the area bounded by the curve $y \geq \sqrt{x}$ & $x > -\sqrt{y}$ & curve $x^2 + y^2 = 2$

Sol. Common region is given by the diagram If area of region OAB = λ then area of OCD = λ

Because $y = \sqrt{x}$ & $x = -\sqrt{y}$

will bound same area with x & y axis respectively.

$y = \sqrt{x}$ $\Rightarrow$ $y^2 = x$

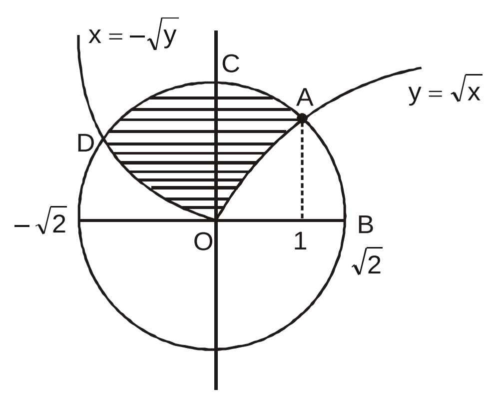
$x = -\sqrt{y}$ $\Rightarrow$ $x^2 = y$ and hence both the curves are

symmetric with respect to line y = x

Area of first quadrant OBC = $\frac{\pi r^2}{4} = \frac{\pi}{2}$ $(\because r = \sqrt{2})$

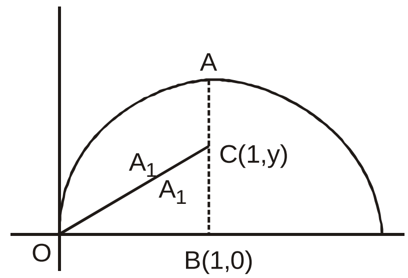
area of region OCA = $\frac{\pi}{2} - \lambda$

area of shaded region = $(\frac{\pi}{2} - \lambda) + \lambda = \frac{\pi}{2}$ sq. units



Ex.10 Find the equation, of line passing through the origin & dividing the curvilinear triangle with vertex at the origin, bounded by the curves $y = 2x - x^2$, $y = 0$ & $x = 1$ into parts of equal area.

Sol. Area of region OBA = $\int_0^1 (2x - x^2)dx = \left[x^2 - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$ sq. units



$\frac{2}{3} = A_1 + A_1 \Rightarrow A_1 = \frac{1}{3}$

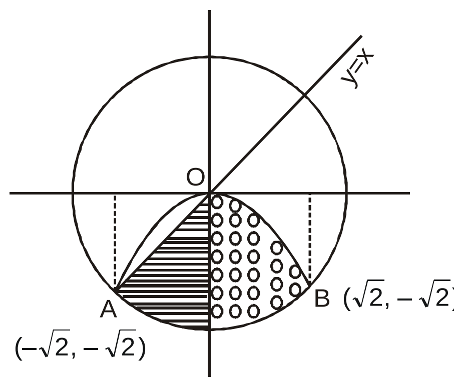
Let point C has coordinates (1, y)

Area of ΔOCB = $\frac{1}{2} \times 1 \times y = \frac{1}{3} \quad \Rightarrow \quad y = \frac{2}{3}$

C has coordinates $\left(1, \frac{2}{3}\right)$; Lines OC has slope $m = \frac{\frac{2}{3} - 0}{1 - 0} = \frac{2}{3}$; Equation of line OC is y = mx $\Rightarrow$ y = $\frac{2}{3}x$

Ex.11 Find the area bounded by the curves $x^2 + y^2 = 4$ & $x^2 = -\sqrt{2}y$ and the line x = y, below x-axis,

Sol. Let C is $x^2 + y^2 = 4$, P is $y = -\frac{x^2}{\sqrt{2}}$ and L is y = x.



We have above three curves. Solving P and C we get the points

$A(-\sqrt{2}, -\sqrt{2})$, $B(\sqrt{2}, -\sqrt{2})$

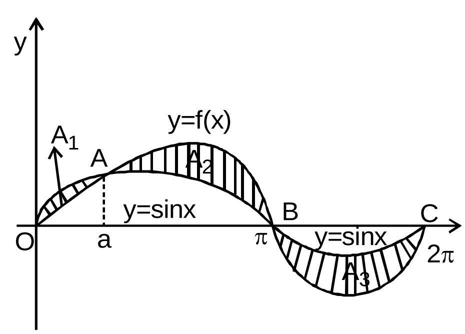
Also the line y = x passes through $A(-\sqrt{2}, -\sqrt{2})$

$\therefore$ Required area = shaded + dotted

$$= \int_{-\sqrt{2}}^{0} (y_3 - y_1)dx + \int_0^{\sqrt{2}} (y_2 - y_1)dx$$

$$= \int_{-\sqrt{2}}^{0} x dx + \int_0^{\sqrt{2}} \frac{-x^2}{\sqrt{2}} dx - \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{4 - x^2} dx = \left[\frac{x^2}{2}\right]_{-\sqrt{2}}^{0} - \frac{1}{\sqrt{2}}\left[\frac{x^3}{3}\right]_0^{\sqrt{2}} - \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-\sqrt{2}}^{\sqrt{2}} \quad \therefore |A| = \frac{3\pi + 16}{6}$$

Ex.12 In the adjacent graphs of two functions y = f(x) and y = sin x are given. y = sinx intersects, y = f(x) at A(a f(a)); B (π, 0) and C(2π, 0). A_i (i = 1, 2, 3) is the area bounded by the curves y = f(x) and y = sinx between x = 0 and x = a, i = 1, between x = a and x = π; i = 2, between x = π and x = 2π; i = 3. If $A_1 = 1 - \sin a + (a - 1) \cos a$, determine the function f(x). Hence determine 'a' and A_1. Also calculate A_2 and A_3.



Sol. From the figure it is clear that $\int_0^a (\sin x - f(x))dx = 1 - \sin a + (a-1)\cos a$ differentiate w.r.t. a

sina – f(a) = – cosa + cosa – (a–1) sin a $\Rightarrow$ sina – f(a) = – a sin a + sin a $\Rightarrow$ f(a) = a sin a $\Rightarrow$ f(x) = x sinx

The points where f(x) & sinx intersect are x sinx = sinx $\Rightarrow$ sinx = 0 or x = 1. We can say that a = 1

$A_1 = \int_0^1 (\sin x - x\sin x)dx = (1 - \sin 1)$ sq. units; $A_2 = \int_1^\pi (f(x) - \sin x)dx = \int_1^\pi (x\sin x - \sin x)dx = (\pi - 1 - \sin 1)$ sq. units

$A_3 = \left|\int_\pi^{2\pi} (\sin x - x\sin x)dx\right| = (3\pi - 2)$ sq. units

Ex.13 The area bounded by y = x^2 + 1 and the tangents to it drawn from the origin is

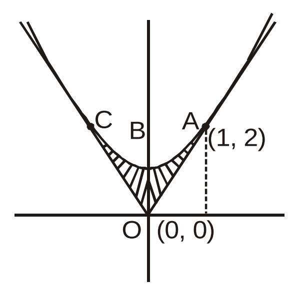
Sol. The parabola is even function & let the equation of tangent is y = mx

Now we calculate the point of intersection of parabola & tangent mx = x^2 + 1

x^2 – mx + 1 = 0 $\Rightarrow$ D = 0 $\Rightarrow$ m^2 – 4 = 0 $\Rightarrow$ m = ± 2

Two tangents are possible y = 2x & y = – 2x

Intersection of y = x^2 + 1 & y = 2x is x = 1 & y = 2



Area of shaded region $OAB = \int_0^1 (y_2 - y_1)\,dx = \int_0^1 ((x^2+1) - 2x)\,dx = \frac{1}{3}$

Area of total shaded region $= 2\left(\frac{1}{3}\right) = \frac{2}{3}$ sq. units